## REFERENCES

1. Tirskii, G. A. and Shevelev, Iu. D., Method of successive approximations for three-dimensional laminar boundary layer problems (locally self-similar case). PMM Vol. 37, № 6, 1973.
2. Shevelev, Iu. D., Difference methods in the calculation of three-dimensional laminar boundary layer. Collection: Some Applications of the Grid Method in Gasdynamics. № 1, 1971.
3. Poots, G., Compressible laminar boundary-layer flow at a point of attachment. J. Fluid Mech., Vol. 22, № 1, 1965.


Translated by J. J. D.

UDC 539.3

# THERMAL STRESSES IN THE PLANE PROBLEM OF THE THEORY OF ELASTICITY CAUSED BY PHASE TRANSFORMATIONS

PMM Vol. 38, № 5, 1974, pp. 847-850

V. I. GOISA
(Moscow)

(Received May 16, 1973)

We study the state of stress in an elastic half-plane in the presence of phase transformations caused by temperature variations at the points of the half-plane. Separately we consider the states of stress caused by the lack of homogeneity in the temperature field and the consequent volume changes taking place in the regions of phase transformations.

Under the term "phase transformation" we understand the structural change in the crystal lattice which occurs when the body is heated above a certain critical temperature [1, 2]. Here the purely thermal stresses are accompanied by the stresses associated with the volume change in the region undergoing phase transformations. Similar problems arise during the investigation of the stress states in the case of elastic tension and in the problems on inclusions. Such problems were studied by D. I. Sherman, Iu. A. Amen-Zade, and others. However in all the problems studied the region occupied by an inclusion was always completely contained within some external region.

The present paper deals with the case in which the boundary separating the media has common points with the outer boundary of the region containing the inclusion. The stresses and strains are assumed to satisfy the conditions of the linear theory of elasticity, with the external region and the inclusion possessing the same elastic properties.

Let us assume that a steady-state plane temperature field is applied to the elastic half-plane $y < 0$ and, that the boundary $y = 0$ is free from external forces. Then the stress components satisfy the following boundary conditions:

$$\sigma_y = \tau_{xy} = 0, \qquad y = 0 \tag{1}$$

The temperature field satisfies the boundary value problem for Laplace equation

$$\Delta T(x, y) = 0 \tag{2}$$
$$T(x, 0) = \begin{cases} T_0, & |x| < a \quad (a = \text{const} > 0) \\ 0, & |x| > a \end{cases}$$

Problem (2) is satisfied by the harmonic function

$$T(x, y) = -\frac{T_0}{\pi}\left(\operatorname{arctg}\frac{a - x}{y} - \operatorname{arctg}\frac{-a - x}{y}\right) \tag{3}$$

We will analyze the thermoelastic stresses according to [3]. In order to solve the problem it is sufficient to determine the Muskhelishvili functions $\Phi(z)$ and $\Psi(z)$, holomorphic in the lower half-plane.

Extending $\Phi(z)$ analytically into the upper half-plane according to the formula

$$\Phi(z) = -\overline{\Phi}(z) - z\,\Phi'(z) - \overline{\Psi}(z), \quad y > 0$$

we express $\Psi(z)$ in terms of $\Phi(z)$

$$\Psi(z) = -\Phi(z) - \overline{\Phi}(z) - z\,\Phi'(z), \quad y < 0.$$

Thus, all the desired quantities can be determined if we know the expression for the function $\Phi(z)$, extended to the entire plane. For the stress components we obtain

$$\sigma_y - i\tau_{xy} = \Phi(z) - \Phi(\bar{z}) + (z - \bar{z})\,\overline{\Phi'(z)} - kT(z, \bar{z}) - k\int \frac{\partial T}{\partial \bar{z}}\,dz \tag{4}$$
$$k = \alpha E / 2(1 - \nu), \quad z = x + iy, \quad \bar{z} = x - iy$$
$$T(z, \bar{z}) = -\frac{T_0}{2\pi i}\left(\ln\frac{a - z}{-a - z} - \ln\frac{a - \bar{z}}{-a - \bar{z}}\right)$$

Substituting this into (1), we obtain the boundary condition for the determination of the function $\Phi(z)$

$$\Phi^+(t) - \Phi^-(t) = -kT(t, \bar{t}) - k\lim_{z\to t}\int \frac{\partial T}{\partial \bar{z}}\,dz, \quad -\infty < t < \infty \tag{5}$$
$$\int \frac{\partial T(z, z)}{\partial \bar{z}}\,dz = -\frac{T_0 z}{2\pi i}\left(\frac{1}{a - \bar{z}} + \frac{1}{a + \bar{z}}\right)$$

The solution of the problem (5) has the form

$$\Phi(z) = -\frac{kT_0}{2\pi i}\left[\ln\frac{a - z}{-a - z} + \frac{1}{2}\left(\frac{a}{z - a} + \frac{a}{z + a}\right)\operatorname{sgn} y\right]$$

Let us compute the stresses. We have

$$\tau_{xy} = -\frac{kT_0 y^2 a}{\pi}\left(\frac{x + a}{|z + a|^4} + \frac{x - a}{|z - a|^4}\right)$$
$$\begin{Bmatrix}\sigma_x\\ \sigma_y\end{Bmatrix} = \frac{kT_0 y a}{2\pi}\left[\pm\frac{y^2 - (x + a)^2}{|z + a|^4} \pm \frac{y^2 - (x - a)^2}{|z + a|^4} - \frac{a}{|z + a|^2} - \frac{a}{|z - a|^2}\right]$$

Obviously, all the stress components vanish at the boundary.

We turn to the second half of the problem. The domain $D$, in which the phase transformation occurs, is bounded from above by the segment $(-a < x < a)$ and from below by the curve $\gamma$, determined from the equation

$$T(x, y) = T_*$$

where ($T_*$ is the phase transformation temperature. After transformation, this equation takes the form $x^2 + (y - b)^2 = r^2 \quad \left(r^2 = a^2 + b^2, b = -\operatorname{ctg} \pi \frac{T_*}{T_0}\right)$

The boundary of the half-plane is free of loads, i. e.

$$\sigma_y = \tau_{xy} = 0, \qquad y = 0 \tag{6}$$

The all around volume expansion can be described with the aid of an additional fictitious applied temperature field

$$\tau(x, y) = \begin{cases} \tau_0, & (x, y) \in D \\ 0, & (x, y) \overline{\in} D \end{cases}, \quad y < 0$$

The points of the domain $D$ tend to undergo the free displacements

$$u_* + iv_* = \alpha\tau_0 z \,/\, 2(\lambda + \mu)$$

Let $u_1 + iv_1$ correspond to the displacements of the points of the domain $D$ and let $u_2 + iv_2$ correspond to the displacements in the remaining part of the half-plane. Then, on the contour $\gamma$ $\quad u_1 + iv_1 = u_2 + iv_2 - \frac{\alpha\tau_0}{2(\lambda + \mu)} t, \quad t \in \gamma \qquad (7)$

In addition, we have the continuity conditions of the normal forces on passing across the curve $\gamma$

$$[X_n + iY_n]_1 = [X_n + iY_n]_2 \tag{8}$$

We assume that the stress components tend to zero at infinity while the displacements remain bounded. Introducing the functions $\varphi_i(z)$ and $\psi_i(z)$ which describe for $i = 1$ the behavior of the material in the domain $D$ and for $i = 2$ the behavior in the remaining part of the lower half-plane, these functions will be holomorphic in $D$ and in the remaining part of the lower half-plane, respectively.

We rewrite the boundary value problem (6)–(8) using the above functions

$$\varphi_i(t) + t\overline{\varphi_i'(t)} + \overline{\psi_i(t)} = 0, \quad -\infty < t < \infty, \quad i = 1, 2 \tag{9}$$

$$\varphi_1(t) + t\overline{\varphi_1'(t)} + \overline{\psi_1(t)} = \varphi_2(t) + t\overline{\varphi_2'(t)} + \overline{\psi_2(t)}, \quad t \in \gamma \tag{10}$$

$$\varkappa\varphi_1(t) - t\overline{\varphi_1'(t)} - \overline{\psi_1(t)} = \varkappa\varphi_2(t) - t\overline{\varphi_2'(t)} - \overline{\psi_2(t)} + g(t) \tag{11}$$

$$\left(g(t) = -A_1 t, \ A_1 = \frac{\alpha\tau_0}{4\mu(\lambda + \mu)}\right)$$

Relation (11), taking into account (10), can be reduced to the following form:

$$(\varkappa + 1)[\varphi_1(t) - \varphi_2(t)] - g(t) \tag{12}$$

If the functions $\varphi_i(z)$ are extended into the regions in the upper half-plane symmetrical with respect to the domain $D$ and to the domain complementary to the lower half-plane, with respect to the abscissae, according to the formula

$$\varphi_i(z) = -z\overline{\varphi_i}'(z) - \overline{\psi}_i(z), \quad y > 0$$

then, we will have

$$\psi_i(z) = -\overline{\varphi}_i(z) - z\varphi_i'(z), \qquad y < 0$$

and condition (10) can be transformed into the form

$$\varphi_1(\bar{t}) - \varphi_2(\bar{t}) = \frac{1}{\varkappa+1} g(t) + \frac{t-\bar{t}}{\varkappa+1} \overline{g'(t)}, \quad t \in \gamma$$

Taking into account that $\varphi_i(\bar{t})$ are the boundary values of the functions $\varphi_i(z)$, holomorphic in the domain $D + \bar{D}$ and the remaining part of the plane, respectively, we can rewrite the previous equality in the following manner:

$$\varphi_1(t) - \varphi_2(t) = \frac{1}{\varkappa+1} g(\bar{t}) + \frac{\bar{t}-t}{\varkappa+1} \overline{g'(\bar{t})}, \quad t \in \bar{\gamma} \tag{13}$$

Combining (12) and (13), we can observe that the function

$$\varphi(z) = \frac{1}{2\pi i(1+\varkappa)} \int\limits_{\gamma} \frac{g(t)\,dt}{t-z} + \frac{1}{2\pi i(1+\varkappa)} \int\limits_{\bar{\gamma}} \frac{g(\bar{t}) + (\bar{t}-t)\overline{g'(\bar{t})}}{t-z}\,dt$$

piecewise-holomorphic in the entire plane, satisfies the conditions (12) and (13) if we assume $\varphi(z) = \varphi_1(z)$ for $z$ inside $\bar{\gamma} + \gamma$ and $\varphi(z) = \varphi_2(z)$ for $z$ situated outside $\gamma + \bar{\gamma}$. In the case under consideration

$$\varphi(z) = -\frac{Az}{2\pi i}\left(\ln_1 \zeta(z) + \ln_2 \zeta(z)\right) + \frac{A}{\pi i}\left[\left(\frac{r^2}{z+ib} + ib\right)\ln_2 \zeta(z) - \frac{r^2}{z+ib}\ln_2 \zeta(-ib), \qquad \zeta(z) = \frac{a-z}{-a-z}\right.$$

In this case, $\ln_i \zeta$ differ from each other by the fact that the lines of discontinuity of the imaginary parts are $\gamma$ and $\bar{\gamma}$, respectively. Differentiating (14), we obtain the expression for $\Phi(z) = \varphi'(z)$.

Knowing the form of $\Phi(z)$, we can write the expressions for the stress components. The form of $\sigma_x$ presents interest since $\sigma_y = \tau_{xy} = 0$ for $y = 0$. We have

$$2\sigma_x = 3\Phi(z) + \overline{3\Phi(z)} + \Phi(\bar{z}) + \bar{\Phi}(z) - (\bar{z} - \bar{z})[\overline{\Phi'(z)} - \Phi'(z)]$$

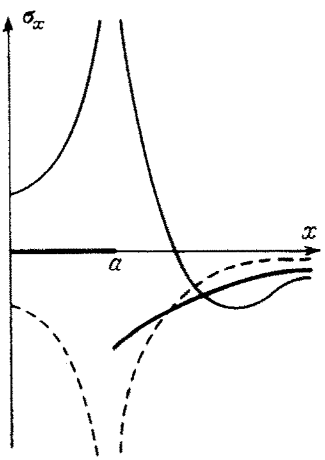


Fig. 1

and at the boundary $y = 0$ we have

$$\sigma_x = \frac{8Ar^2b}{\pi}\frac{x}{(x^2+b^2)^2}\ln\left|\frac{a-x}{a+x}\right| - AB\frac{C-x^2}{(x^2+b^2)^2}$$

$$B = \begin{cases} B_1, & |x| < a \\ B_2, & |x| > a \end{cases}, \qquad C = \begin{cases} C_1, & |x| < a \\ C_2, & |x| > a \end{cases}$$

$$B_1 = 4r^2\left(1 + \frac{1}{\pi i}\ln\frac{a+ib}{a+ib}\right), \qquad B_2 = B_1 - 4r^2$$

$$C_1 = \frac{B_1 8b/\pi}{B_1 + 8b/\pi}, \qquad C_2 = \frac{\pi i B_2 - 8ib/\pi}{\pi i B_2 + 8ib/\pi}$$

Here, $B_1$, $B_2$, $C_1$, $C_2$ are real constants.

The form of the curves $y = \sigma_x(x)$ for $x > 0$ are represented in Fig. 1 by thin lines for $b = -1$, by heavy lines for $b = 0$ and by broken lines for $b = +1$. The component $\sigma_x$ of the stress tensor as a function of $x$ and $y$ has singularities at the boundary of the half-plane at the points $y = 0$, $x = \pm a$. When the temperature $T_0$ increases, these singularities give rise to infinite tensile stresses which cause cracking of the material near these singularities. In the real body plastic zones appear near these points and the stresses in these zones are redistributed and smoothed out. The singularities are caused by the fact that the contour $\gamma$ reaches the free boundary and they represent a case different from the case in which the region occupied by the inclusion is internal.

REFERENCES

1. Physical Metallurgy (Edited by R. Kan). Ed. 2, Moscow, "Mir", 1967.
2. Lozinskii, M. G., The structure and the properties of metals at high temperatures. Moscow, Metallurgizdat, 1963.
3. Filipkowski, I., Dwuwymiarowe zagadnienie teorii naprężeń cieplnych. Warszawa, Rozprawy inżynierskie, Vol. 14, № 4, 1966.

Translated by E. D.

UDC 539.3

## ON THE PROOF OF THE SAINT-VENANT PRINCIPLE FOR BODIES OF ARBITRARY SHAPE

PMM Vol. 38, № 5, 1974, pp. 851-864
V. L. BERDICHEVSKII
(Moscow)
(Received January 30, 1974)

Toupin [1] proved that the stresses in a cylindrical rod, caused by application of a self-equalizing load at the endface, decrease exponentially with distance from the endface. An estimate has been obtained for a constant in the exponential in terms of the smallest natural vibrations frequency of an elastic cylinder.

A determination of the energy decay rate is given below for bodies of arbitrary shape and its estimate is given in terms of some characteristics of the body geometry, including the Poincaré and Korn constants of the cross section. These constants are known in the case of a circular rod and the estimate is given in numbers.

The dependence of the energy decay rate on the body shape is examined. It is shown that for cone-type bodies a power-law estimate holds for the energy decay which goes into an exponential estimate in the limit as the cone degenerates into a cylinder. Analogous estimates for the stresses result from the estimates for the energy.

**1. Determination of the energy decay rate.** Within the framework of a geometrically linear theory, let us generally consider an inhomogeneous, anisotropic and physically nonlinear elastic solid (see [2]). We refer the undeformed state of the solid to a Cartesian coordinate system $x^\circ \equiv x, x^\alpha$ (the Greek superscripts $\alpha$, $\beta$, $\gamma$, ... take on the values 1, 2).

Let the part of the solid in the half-plane $x > 0$ be load-free and let the state of stress be caused by some external effects on the part of the solid in the half-plane $x \leqslant 0$. Further we will study the parameters independent of the nature of these effects, therefore without limiting the generality, it can be assumed that the deformation of the body in the $x \geqslant 0$ half-plane is caused by some surface forces applied in a section of the solid by the $x = 0$ plane.